Do integrable cellular automata have the confinement property?

2007 J. Phys. A: Math. Theor. 40 F725

(http://iopscience.iop.org/1751-8121/40/30/F05)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) F725–F735 doi:10.1088/1751-8113/40/30/F05

FAST TRACK COMMUNICATION

Do integrable cellular automata have the confinement property?

B Grammaticos[1], **A Ramani**[2], **K M Tamizhmani**[3], **T Tamizhmani**[4] **and A S Carstea**[5]

[1] IMNC, Université Paris VII-Paris XI, CNRS, UMR 8165, Bât. 104, 91406 Orsay, France
[2] Centre de Physique Théorique, Ecole Polytechnique, CNRS, 91128 Palaiseau, France
[3] Departement of Mathematics, Pondicherry University, Kalapet, 605014 Puducherry, India
[4] Department of Mathematics, Kanchi Mamunivar Centre for Postgraduate Studies, Puducherry, India
[5] Department of Theoretical Physics, Institute of Physics and Nuclear Engineering, 407 Atomistilor, Magurele, 077125 Bucharest, Romania

Received 29 May 2007, in final form 13 June 2007
Published 12 July 2007
Online at stacks.iop.org/JPhysA/40/F725

Abstract
We analyse a criterion, introduced by Joshi and Lafortune, for the integrability of cellular automata obtained from discrete systems through the ultradiscretization procedure. We show that while this criterion can be used in order to single out integrable ultradiscrete systems, there do exist cases where the system is nonintegrable and still the criterion is satisfied. Conversely we show that for ultradiscrete systems that are derived from linearizable mappings the criterion is not satisfied. We investigate this phenomenon further in the case of a mapping which includes a linearizable subcase and show how the violation of the criterion comes to be. Finally, we comment on the growth properties of ultradiscrete systems.

PACS numbers: 02.30.Ik, 02.30.Ks

1. Introduction

Integrability detectors are important because integrable systems are both interesting and rare. Of course, when one uses a constructive method for the derivation of a new integrable system, the use of an integrability detector is not an imperative. At best, in this case, the implementation of an integrability test can be used in order to gauge the efficiency of the detector itself. However, these 'constructive' cases are not the most challenging ones. What is far more interesting is the situation when one derives a model, usually based on physical arguments, and wonders about its possible integrable character.

In the domain of differential equations the Painlevé property is the indisputable integrability detector [1]. It is based on the requirement that the solutions of a given equation be devoid of multivaluedness-inducing singularities. In this sense, the Painlevé property becomes almost tautologically identified to integrability (since the absence of multivaluedness allows one to integrate the equations in the Poincaré sense). Efficient algorithms for the investigation of the Painlevé property in differential equations have been proposed already by the founders of the approach (Kowalevskaya, Painlevé, Gambier, . . .). More recently Ablowitz, Ramani and Segur (ARS) [2] have introduced their algorithm which has played a major role in the development of the modern era of integrability studies. (Still one has to keep in mind that, efficient though it is, the ARS algorithm is not perfect and can only detect a certain type of multivaluedness, as stressed by Kruskal [3].) Obviously, an integrability detector is related to a specific type of integrability, of which there exist several kinds, the term integrability being conveniently rather vague. The Painlevé property is thus characteristic of systems the integration of which proceeds through spectral methods. Typical examples are the Painlevé equations, to say nothing of the slew of integrable evolution equations discovered over the past quarter century. However, there exists a large class of systems the integrability of which is obtained in a simpler way. They are the linearizable, or c-integrable in the Calogero [4] terminology, systems. As we have shown in [5], their integrable character is not associated with the Painlevé property. (As a matter of fact, no linearizability detector appears to exist to date, to the authors knowledge.)

Discrete systems pose a greater challenge. A first integrability detector was proposed based on the observation that mappings integrable through spectral methods have confined singularities [6], i.e., any singularity spontaneously appearing due to the choice of initial conditions disappears after a few iteration steps. While singularity confinement has been instrumental in discovering a host of integrable discrete systems, it turned out that the confinement property was not sufficient in order to guarantee integrability. We shall not go into detailed explanations here. It suffices to say that for discrete systems to be integrable, a proper local singularity structure is not enough. The growth properties of the solutions at infinity enter into play and the best way to qualify this is through the Nevanlinna approach [7]. To put it in a nutshell, for a discrete system to be integrable the requirement is that the Nevanlinna order of the solution be finite (which guarantees not too fast a growth) and moreover that its singularities be confined. As in the continuous case, linearizable discrete systems are a class of their own. As we have shown in [5] linearizability does not require confined singularities although the solutions must still have finite Nevanlinna order. An algorithm which calculates the growth was proposed by Hietarinta and Viallet [8] and is commonly referred to as the algebraic entropy technique.

It is thus natural at this point to ask how the integrability-related properties of discrete systems carry over to cellular automata obtained from discrete systems following the ultradiscretization procedure [9]. We recall here that the latter consists into introducing an ansatz $x = \mathrm{e}^{X/\delta}$ (where x is the solution of the discrete system, which should obviously be positive definite) and obtain for X an equation by going to the limit $\delta \to 0$. The essential identity that allows to us derive easily the ultradiscrete forms is $\lim_{\delta\to 0} \delta \log(\mathrm{e}^{A/\delta} + \mathrm{e}^{B/\delta}) = \max(A, B)$. The ultradiscretization procedure *preserves any integrable character* of the initial system. One would thus naturally expect the ultradiscrete analogue of integrability-related properties, like the singularity confinement of the discrete case, to exist. This would allow one to formulate ultradiscrete integrability conjectures and propose integrability detectors. This question has been already addressed by Joshi and Lafortune [10] who proposed a singularity analysis approach which is perceived as the ultradiscrete equivalent of singularity confinement. In this paper, we shall critically examine this approach and show that the situation is more complicated than what one would initially expect. In particular, we shall show that, just as in the discrete case, there exist integrable ultradiscrete systems with unconfined singularities but also nonintegrable systems with confined singularities.

2. Singularities and their confinement

Before proceeding to the analysis of ultradiscrete systems it is interesting to spend a few lines on their discrete counterparts focusing on the notion of singularity. Given a mapping of the form $x_{n+1} = f(x_n, x_{n-1})$ we are in the presence of a singularity whenever $\frac{\partial x_{n+1}}{\partial x_{n-1}} = 0$, i.e., x_{n+1} 'loses' its dependence on x_{n-1}. When this is due to a particular choice of initial conditions we are referring to this singularity as a *movable* one. Movable singularities may be bad, for integrability, because they may lead, after a few mapping iterations, to an indeterminate form $(0/0, \infty - \infty, \ldots)$ or propagate indefinitely. In the former case, provided we can lift the indeterminacy while recovering the lost degree of freedom (using an argument of continuity with respect to the initial conditions), we are talking about a confined singularity. As explained in the introduction, mappings which are integrable through spectral methods have confined singularities. The typical singularity pattern in this case is the following: the solution is regular for all values of the index n up to some value n_s, then a singularity appears and propagates up to n_c whereupon it disappears and the solution is again regular for all values of the index larger than n_c. In some cases we are in the presence of the reciprocal situation. The solution is singular for all values of $n < n_s$, becomes regular between n_s and n_c and is again singular for $n > n_c$. This singularity is called weakly confined by Takenawa [11] and is considered to be compatible with integrability. At the limit where there exists no interval where the solution may be regular, and the solution is singular throughout, we are in the presence of what we call a 'fixed' singularity (which again does not hinder integrability).

How can these notions be transposed to the ultradiscrete setting? This is a question that has been addressed by Joshi and Lafortune [10] who proposed an analogue to the singularity confinement property for ultradiscrete mappings. In the ultradiscrete systems, the nonlinearity is mediated by terms involving the max operator. Typically, one is in the presence of terms like $\max(X_n, 0)$. When, depending on the initial conditions, the value of X_n crosses zero, the result of the $\max(X_n, 0)$ operation becomes discontinuous: when X is slightly smaller than 0 the result is zero, while for $X > 0$ the result is X. It is this discontinuity that plays the role of the singularity. Typically, if we put $X = \epsilon$, a term $\mu = \max(\epsilon, 0)$ propagates with the iterations of the mapping and perpetuates the discontinuity *unless* by some coincidence it disappears. This disappearance is the equivalent of the singularity confinement for ultradiscrete systems. Joshi and Lafortune [10] have introduced an algorithmic method for testing the confinement property for ultradiscrete systems, linked it to integrability and reproduced results on ultradiscrete Painlevé equations by suitably deautonomising ultradiscrete mappings.

Before proceeding to a critical analysis of the method let us give an illustrative example. In [12] we have introduced three different forms for the ultradiscrete Painlevé I equations starting from the QRT mapping

$$x_{n+1}x_{n-1} = a\frac{1 + x_n}{x_n^{\sigma}} \qquad \sigma = 0, 1, 2 \tag{2.1}$$

and considering its nonautonomous form. In order to illustrate the singularity analysis approach we shall limit ourselves to the autonomous case and moreover take $\sigma = 2$. Ultradiscretizing (2.1) (putting $x = \mathrm{e}^{X/\delta}$, $a = \mathrm{e}^{A/\delta}$ and taking $\delta \to 0$) we find

$$X_{n+1} + X_{n-1} = A + \max(0, X_n) - 2X_n. \tag{2.2}$$

The singularity corresponds to the discontinuity induced by the term $\max(0, X_n)$ when the value of X_n crosses zero. We shall thus examine the behaviour of a singularity appearing at, say, $n = 1$ where $X_1 = \epsilon$, while X_0 is regular and look at the propagation of this singularity both forwards and backwards. In what follows, we introduce the notation $\mu \equiv \max(\epsilon, 0)$ and the presence of μ indicates that the value of X is singular. Below we present only the results corresponding to $A > 0$, those corresponding to $A < 0$ leading to similar conclusions. First we examine the case $X_0 < 0$ and $|X_0| < A$ where one can see a regular zone between X_{-3} and X_1 and a singular pattern from X_2 on as well as until X_{-4}.

$$
\begin{aligned}
&\vdots\\
&X_{-13} = X_{-7} - 2X_{-5}\\
&X_{-12} = X_{-6} - 2X_{-5}\\
&X_{-11} = X_{-5}\\
&X_{-10} = X_{-7} - X_{-5}\\
&X_{-9} = X_{-6} - X_{-5}\\
&X_{-8} = X_{-5}\\
&X_{-7} = A + \epsilon\\
&X_{-6} = -X_0 - 2\epsilon + \mu\\
&X_{-5} = X_0 + \epsilon - \mu\\
&X_{-4} = A - X_0 - \epsilon + \mu\\
&X_{-3} = -\epsilon\\
&X_{-2} = X_0 + \epsilon\\
&X_{-1} = A - 2X_0 - \epsilon\\
&X_0\\
&X_1 = \epsilon\\
&X_2 = A - X_0 - 2\epsilon + \mu\\
&X_3 = X_0 + \epsilon - \mu\\
&X_4 = -X_0 + \mu\\
&X_5 = A - \epsilon\\
&X_6 = X_3\\
&X_7 = X_4 - X_3\\
&X_8 = X_5 + X_3\\
&X_9 = X_3\\
&X_{10} = X_4 - 2X_3\\
&X_{11} = X_5 + 2X_3\\
&\vdots
\end{aligned}
$$

This is a weakly confined case, in the sense that a (small) regular region exists surrounded by singular values extending all the way to infinity in both directions. As we explained already such a behaviour is deemed compatible with integrability. The cases $0 < X_0 < A$ and $X_0 < -A$ lead to similar, weakly confined, patterns. The last case is $X_0 > A$ where the solution is regular until X_1 then singular, confined, between X_2 and X_4 and regular from X_5 on.

$$\begin{aligned}
&\vdots\\
&X_{-3} = A - \epsilon\\
&X_{-2} = X_0 - A + 2\epsilon\\
&X_{-1} = -X_0 + A - \epsilon\\
&X_0 = X_0\\
&X_1 = \epsilon\\
&X_2 = A - X_0 - 2\epsilon + \mu\\
&X_3 = 2X_0 - A + 3\epsilon - 2\mu\\
&X_4 = A - X_0 - \epsilon + \mu\\
&X_5 = -\epsilon\\
&X_6 = X_0 + 2\epsilon\\
&\vdots
\end{aligned}$$

Thus, in all cases we have either a confined singularity (a central singular zone with regular behaviour outside) or a weakly confined singularity (a central regular zone with singular behaviour outside). Both behaviours are deemed compatible with integrability. The two points which we consider important in this analysis are that (a) one must study all possible sectors of initial conditions and/or parameters and (b) one must consider the possibility of weakly confined solutions.

3. Nonintegrable systems with confined singularities and integrable systems with unconfined singularities

As we explained in the introduction there exist discrete systems which while being nonintegrable still possess confined singularities. This discovery has as a consequence that singularity confinement alone cannot be used as a discrete integrability detector. As we shall show now the same problem appears in an ultradiscrete setting. In [13] we obtained a mapping which did pass the confinement test while having a positive algebraic entropy

$$x_{n+1} = x_{n-1}\left(x_n + \frac{1}{x_n}\right). \tag{3.1}$$

The main advantage of this mapping over the examples of [8] is that it is multiplicative and by choosing the appropriate initial data one can restrict the solution to positive values. In that case the ultradiscretization of (3.1) is straightforward. We find

$$X_{n+1} = X_{n-1} + |X_n| \tag{3.2}$$

where we have preferred to introduce the absolute value of X instead of its equivalent $\max(X, 0) + \max(-X, 0)$. We shall examine the behaviour of a singularity appearing at, say, $n = 1$ where $X_1 = \epsilon$, while X_0 is regular. We again use the identity $\mu \equiv \max(\epsilon, 0) = (|\epsilon| + \epsilon)/2$ and distinguish two different sectors $X_0 < 0$ and $X_0 > 0$. In the first case ($X_0 < 0$) we find the sequence

$$\begin{aligned}
&\vdots\\
&X_{-3} = 3X_0\\
&X_{-2} = 2X_0 - \epsilon\\
&X_{-1} = X_0 + \epsilon
\end{aligned}$$

$$
\begin{aligned}
&X_0\\
&X_1 = \epsilon\\
&X_2 = X_0 - \epsilon + 2\mu\\
&X_3 = -X_0 + 2\epsilon - 2\mu\\
&X_4 = \epsilon\\
&X_5 = -X_0 + \epsilon\\
&\vdots
\end{aligned}
$$

We can see readily that the singularity, indicated by the presence of μ, is confined (to X_2 and X_3 only). Turning to the case $X_0 > 0$ we find the sequence

$$
\begin{aligned}
&\vdots\\
&X_{-4} = -X_0 + 2\mu + \epsilon\\
&X_{-3} = -X_0 + 2\mu\\
&X_{-2} = \epsilon\\
&X_{-1} = -X_0 + \epsilon\\
&X_0\\
&X_1 = \epsilon\\
&X_2 = X_0 + 2\mu - \epsilon\\
&X_3 = -X_0 + 2\mu\\
&X_4 = 2X_0 + 4\mu - \epsilon\\
&\vdots
\end{aligned}
$$

In this case, we are in the presence of a weakly confined solution: a regular part around $n = 0$ is surrounded by unconfined singularities both for large positive and large negative n's. Thus, the ultradiscrete mapping (3.2) has confined singularities and is not integrable. (A stronger indication concerning this nonintegrability, based on growth properties, rather than the analogy with the discrete case, will be presented in section 5.) In this sense, system (3.2) is an ultradiscrete analogue of the equation discovered by Hietarinta and Viallet [8].

The converse situation, of a mapping which, while integrable, does not possess confined singularities does also exist. As expected an example is to be sought among linearizable systems. In [13] we discovered the 'multiplicative' linearizable mapping

$$\frac{x_{n+1}}{x_{n-1}} = a\frac{x_n + a}{x_n + 1}. \tag{3.3}$$

It is straightforward to check that the parameter a can be always taken larger than unity. (Indeed it suffices to reverse the direction of the evolution in which case a goes to $1/a$.) We can now ultradiscretize (3.3) to

$$X_{n+1} = X_{n-1} + A + \max(X_n, A) - \max(X_n, 0) \tag{3.4}$$

where $A > 0$. The complete description of the solution would require examining several sectors which exist but in order to show that there exist unconfined singularities it suffices to exhibit such a situation in one sector. It turns out that the case where X_0 has a large negative value is one leading to unconfined singularities:

$$\begin{aligned}
&\vdots\\
&X_{-4} = -X_0 - 4A\\
&X_{-3} = -4A + \epsilon\\
&X_{-2} = X_0 - 2A\\
&X_{-1} = -2A + \epsilon\\
&X_0\\
&X_1 = \epsilon\\
&X_2 = X_0 + 2A - \mu\\
&X_3 = 2A + \epsilon\\
&X_4 = X_0 + 3A - \mu\\
&X_5 = 4A + \epsilon\\
&X_6 = X_0 + 4A - \mu\\
&X_7 = 6A + \epsilon\\
&\vdots
\end{aligned}$$

We remark readily that while for negative indices the solution is regular, a singularity, mediated by μ, appears for positive n's and is never confined. In section 5, we will analyse mapping (3.4) from the point of view of the growth of the solutions.

Thus in perfect parallel to the discrete situation there exist ultradiscrete systems where despite the nonintegrable character we have confined singularities while for ultradiscrete systems obtained from linearizable mappings the singularities are not confined.

4. A family of integrable mappings and their ultradiscrete counterparts

In this section, we shall pursue the study of the singularities of ultradiscrete systems which come as limits of mappings of the QRT family [14] and discuss their special properties. In particular, we shall examine a mapping of the form

$$(x_{n+1}x_n - 1)(x_n x_{n-1} - 1) = \frac{x_n^4 + a x_n^2 + 1}{(1 + x_n/b)^\sigma} \qquad \sigma = 0, 1, 2. \tag{4.1}$$

Mapping (4.1) is a special subcase of the autonomous limit of q-discrete Painlevé V. When $\sigma = 0$ the mapping was shown in [15] to be linearizable. All three cases belong to the QRT family and do possess a conserved quantity. We introduce $y_n = x_{n+1}x_n - 1$ and (with obvious notations) we obtain the ultradiscrete form of (4.1)

$$\begin{aligned}
X_{n+1} &= -X_n + \max(Y_n, 0)\\
Y_n &= -Y_{n-1} + \max(4X_n, 2X_n + A, 0) - \sigma \max(X_n - B, 0).
\end{aligned} \tag{4.2}$$

Let us concentrate first on the $\sigma = 0$ case. The singularity corresponds here to the value of Y crossing 0. We thus put $Y_0 = \epsilon$ and iterate (4.2) starting from X_0 both backwards and forwards. We examine the branch $0 < X_0 < A/2$. This is the sequence we find for $n < 0$:

$$X_n = X_0 + n(A - \epsilon) \qquad Y_n = X_n + X_{n+1}. \tag{4.3}$$

At $n = 0$ we have by definition X_0 and $Y_0 = \epsilon$. At $n = 1$ we find a singular value

$$X_1 = -X_0 + \mu \tag{4.4}$$

and iterating for positive n we obtain

$$X_{n+1} = X_1 + n(A - \epsilon) \qquad Y_n = X_n + X_{n+1}. \tag{4.5}$$

Since X_{n+1} contains X_1, the singularity which appeared at $n = 1$ propagates *ad infinitum*. On the other hand, since (4.1) with $\sigma = 0$ is a member of the QRT family it does have an invariant:

$$K = \frac{x_n^2 + x_{n-1}^2 + a}{x_n x_{n-1} - 1}. \tag{4.6}$$

Ultradiscretizing (4.6) is straightforward

$$K = \max(4X, 2X + A, 2\max(Y, 0)) - 2X - Y. \tag{4.7}$$

We can check that (4.7) is indeed conserved by (4.2) and at no point does the singularity hinder this conservation.

Thus, we are here in the presence of an integrable mapping with unconfined singularities. This counterexample to the integrability criterion of [10] is even more serious than the examples of section 3 since the mapping here possesses an explicit invariant. It is thus natural to wonder what does happen in the remaining cases of (4.2), $\sigma = 1$ and 2. Presenting exhaustive results, as in the case of section 2, would be prohibitively long. Below we present a few typical numerical examples. We start with the case $\sigma = 2$, take parameters $A = 100$ and $B = 11$, and initial condition $X_0 = 7$. We obtain the sequence:

$$\begin{aligned}
&\vdots\\
&X_{-3} = -15 + \epsilon - \mu\\
&Y_{-3} = \epsilon\\
&X_{-2} = 15\\
&Y_{-2} = 122\\
&X_{-1} = 107\\
&Y_{-1} = 114\\
&X_0 = X_0\\
&Y_0 = \epsilon\\
&X_1 = -7 + \mu\\
&Y_1 = 86 - \epsilon + 2\mu\\
&X_2 = 93 - \epsilon + \mu\\
&Y_2 = 122 - \epsilon\\
&X_3 = 29 - \epsilon + \mu\\
&Y_3 = \epsilon\\
&X_4 = -29 + 2\mu\\
&Y_4 = 42 + 3\epsilon - 4\mu\\
&\vdots
\end{aligned}$$

We remark that this is a weakly confined singularity. A regular pattern exists between Y_{-3} and Y_0 and the singularity extends all the way to $\pm\infty$ on the outside. What is more interesting is that the value of Y comes backs to zero, up to a quantity of $\mathcal{O}(\epsilon)$, repeatedly albeit not in a periodic way. As a matter of fact the values of n for which Y is of order ϵ do show some regularity: $\ldots, -26, -22, -19, -16, -13, -10, -6, -3, 0, 3, 7, 10, 13, 16, 19, 23, 26, \ldots$. We remark that the interval between two successive near-zeros is either 3 or 4 but as far as we can tell there is no particular regularity in the succession of these two numbers. Similar results can be obtained in the $\sigma = 1$ case. Again we find a weakly confined singularity and the near-zeros of Y appear at values $\ldots, -34, -29, -25, -21, -17, -13, -8, -4, 0, 4, 9, 13, 17, 21, 25, 30, 34, \ldots$. By studying the variation of the (mean) length of the intervals between two successive near-zeros of Y, which, we point out again here, also give the length of the regular zone, we arrive at the following conclusion. For fixed (appropriate) values of X_0 and A and increasing values of B, with $2B/A$ integer, the length is exactly $2B/A + 3$ for $\sigma = 2$ and $2B/A + 4$ for $\sigma = 1$. If Y_0 takes exactly the value 0 then the solution is strictly periodic. If $2B/A$ is not integer then these quantities give the mean length of the interval. We can now see what is happening in the $\sigma = 0$ case. We can obtain this case by starting from $\sigma = 1$ or 2 and take $B \to \infty$. Thus at the limit the length of the regular zone becomes infinite and we go from a situation of weakly confined singularities to one of an unconfined singularity.

At this point one can wonder what is happening in the case where the mapping is not integrable. We take (4.2) with $\sigma = 3$ and choose the same parameters as for the case analysed just above, namely $A = 100$, $B = 11$ with initial conditions $X_0 = 7$ and Y_0 going through zero. Iterating the mapping we find that the solution does *not* recur to $\mathcal{O}(\epsilon)$ although it does repeatedly cross zero to change sign. So for negative n the solution is regular while for positive values of n the singularity continues indefinitely. Thus in this case we have unsurprisingly an unconfined singularity.

In our analysis above we have presented the 'interesting' singularity patterns. There also exist ranges of parameters in combination with the initial value X_0 for which the solution has strictly confined singularities. Their study does not bring any new element: it suffices that *one* unconfined singularity pattern exist for confinement to be violated.

5. Growth properties of ultradiscrete systems

As we have seen in the previous sections, the situation concerning the integrability criterion of [10] is far from clear. Counterexamples exist both as to its sufficient and as to its necessary character. This does not mean that the criterion is not useful. As was shown by Joshi and Lafortune there exist many instances where the criterion can be put to use and successfully predict integrable deautonomizations. Still, because of the counterexamples, one is tempted to look for auxiliary or complementary criteria. Since in the discrete case growth arguments turned out to be crucial for integrability it makes sense to try to adapt these arguments to the case of ultradiscrete systems.

Clearly, the complexity argument used in the case of discrete systems (and its implementation through the algebraic entropy techniques) cannot be transposed as such to the ultradiscrete case. Still the growth of the values of the variable can be of interest as we shall see in what follows.

We start with the integrable ultradiscrete system (2.2) and iterate it backwards and forwards for parameter $A = 7$ and initial values $X_0 = -100$ and $X_1 = 0$. We find the following sequence of values: $\ldots, -100, 107, 0, -100, 207, -100, 0, 107, -100, 100, 7, -100, 200, -93, -7, 114, -100, 193, -86, -14, 121, -100, 86, 21, \ldots$. We remark that the solution does not grow but oscillates around zero. As a matter of fact the absolute value of the solution never exceeds the value $2|X_0| + |A|$. Similar results can be obtained for other values of the parameter and initial conditions. Another integrable ultradiscrete system with an explicit conserved quantity is (4.2). In section 4, we have given numerical values of the iterates of the case $\sigma = 2$, with parameters $A = 100$, $B = 11$ and initial condition $X_0 = 7$. Again the solution is not growing but bouncing between values which in this case never exceed $2B + A$.

It may turn out that this property of bounded, bouncing solution is characteristic of a certain class of integrable ultradiscrete systems. Clearly, more detailed studies are needed before one can make a more affirmative statement. What is clear at this stage is that *not all* integrable ultradiscrete systems do have such solutions. Analysing the growth of (4.2) with $\sigma = 0$ (which in the discrete case is not just QRT integrable but in fact linearizable) we find the sequence of values, for $A = 100$ and $X_0 = 7$, $Y_0 = 0$. We have for X: $\ldots, 207, 107, 7, -7, 93, 193, 293, \ldots$ and values that grow linearly by steps of 100 away from zero in both positive and negative directions. Similarly for Y we find $\ldots, 314, 114, 0, 86, 286, \ldots$ and linear growth in steps of 200 away from zero in both directions. In order to investigate whether this linear growth is a property of ultradiscrete systems coming from linearizable mappings we analyse the solutions of (3.4), taking $A = 10$, $X_0 = 0$ and $X_1 = 7$. We find the sequence $\ldots -60, -53, -40, -33, -20, -16, 0, 7, 13, 17, 23, 27, 33, 37, \ldots$. Again we have a linear growth of the solution. For negative n the solution is increasing with alternating steps of 7 and 13 while for positive n we have alternating steps of 4 and 6. Another example can be given by the mapping

$$X_{n+1} = -X_{n-1} + X_n + \max(X_n, 0) \tag{5.1}$$

which comes from the linearizable discrete system $x_{n+1}x_{n-1} = x_n(x_n + 1)$. Again starting from initial conditions $X_0 = 0$ and $X_1 = 1$ we find $X_n = n$, obviously a linear growth.

While integrable mappings have moderate growth nonintegrable ones like (3.2) may grow much faster. By inspection we conclude that the solutions of (3.2) form a Fibonacci sequence and thus grow exponentially fast. On the other hand, exponential growth is not the only possible one. For instance if we consider the ultradiscrete analogue of (2.1) with $\sigma = -1$, which is not integrable, we find that the growth of the solutions is quadratic. What is making the situation even more complicated is for (4.2) with $\sigma = 3$, which is clearly a nonintegrable case, we find a bounded, bouncing solution.

In view of the above here are the (few) conclusions one can draw with respect to growth properties of ultradiscrete systems. If one finds an exponential growth of the values of the iterates this is an indication of nonintegrability, while a linear growth indicates linearizability. However one must bear in mind the fact that even in these cases a slower growth may be possible. Thus, the growth properties for ultradiscrete systems can be of some assistance in the detection of integrability but they do not constitute a powerful tool as in the discrete case.

6. Conclusion

In this paper, we have investigated an integrability criterion for ultradiscrete systems introduced by Joshi and Lafortune. They based their criterion in the disappearance of discontinuities induced by terms like $\max(X, 0)$ when X crosses zero. Adopting the terminology used for discrete systems this discontinuity is dubbed a singularity and its disappearance is the equivalent of the confinement of the singularity.

A perfect criterion of integrability would in theory be both necessary and sufficient. However no real-life integrability criterion meets these stringent requirements. This has also to do with the loose definition of 'integrability' which is some cases is used in lieu of 'linearizability' or even 'solvability'. So, expectedly, it turns out that the criterion proposed by Joshi and Lafortune while being efficient in many instances is not perfectly failsafe. As we have shown here there exist ultradiscrete systems with confined singularities and which are nonintegrable. Conversely some systems integrable through linearization do not have confined singularities, in perfect parallel to the discrete situation. What is more worrisome here is that there exists at least one example of a linearizable system which possesses an explicit invariant and still has unconfined singularities.

We have also looked at the growth properties of the iterates of mappings in order to complement the singularity analysis. The situation is not as clear as in the discrete case. While exponential growth is an indication of nonintegrability there exist nonintegrable ultradiscrete systems with growth less than exponential. As usual linearizable systems are a class on their own, with linear growth of the values signalling linearizability.

Thus, the answer the question of the title of this paper is a qualified 'yes' but exceptions do exist, just as in the case of integrable mappings which do not always possess the confinement (discrete Painlevé) property.

References

[1] Ramani A, Grammaticos B and Bountis T 1989 *Phys. Rep.* **180** 159
[2] Ablowitz M J, Ramani A and Segur H 1978 *Lett. Nuovo Cimento* **23** 333
[3] Kruskal M D 1992 *Flexibility in Applying the Painlevé Test (NATO ASI B278)* (New York: Plenum) p 187
[4] Calogeron F 1990 *What is Integrability?* ed V Zakharov (New York: Springer) p 1
[5] Ramani A, Grammaticos B and Tremblay S 2000 *J. Phys. A: Math. Gen.* **33** 304
[6] Grammaticos B, Ramani A and Papageorgiou V 1991 *Phys. Rev. Lett.* **67** 1825
[7] Ablowitz M J, Halburd R and Herbst B 2000 *Nonlinearity* **13** 889
Grammaticos B, Tamizhmani T, Ramani A and Tamizhmani K M 2001 *J. Phys. A: Math. Gen.* **34** 3811
[8] Hietarinta J and Viallet C M 1991 *Phys. Rev. Lett.* **81** 325
[9] Tokihiro T, Takahashi D, Matsukidaira J and Satsuma J 1996 *Phys. Rev. Lett.* **76** 3247
[10] Joshi N and Lafortune S 2005 *J. Phys. A: Math. Gen.* **38** L499
Joshi N and Lafortune S 2006 *Nonlinearity* **19** 1295
[11] Takenawa T 2001 *J. Phys. A: Math. Gen.* **34** L95
[12] Grammaticos B, Ohta Y, Ramani A and Takahashi D 1998 *Physica* D **114** 185
[13] Tsuda T, Ramani A, Grammaticos B and Takenawa T 2007 A class of integrable and nonintegrable mappings and their dynamics *Preprint*
[14] Quispel G R W, Roberts J A G and Thompson C J 1989 *Physica* D **34** 183
[15] Joshi N, Grammaticos B, Tamizhmani T and Ramani A 2006 *Lett. Math. Phys.* **78** 27